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Generalised Spin Structures

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Idea

Let M be a spin manifold.

spin structure $\rightsquigarrow$ spinor bundle $\rightsquigarrow$ special spinors $\rightsquigarrow$ geometric properties

Example

If M admits a spin structure carrying a nowhere-vanishing parallel spinor, then M is Ricci-flat.

- **Question**: what if M is not spin?
- **Idea**: equip every orientable manifold with spin-like structures.

Spin structures I

Let M^n be an oriented Riemannian manifold

with bundle of oriented orthonormal frames FM.

A **spin structure** is a lift of the structure group of FM to the group $\mathrm{Spin}(n)$ along the double covering

$$\lambda_n \colon \mathrm{Spin}(n) \to \mathrm{SO}(n).$$

In other words, it is a pair (P, Φ) where

- P is a principal $\mathrm{Spin}(n)$-bundle over M, and
- $\Phi \colon P \to FM$ is a $\mathrm{Spin}(n)$-equivariant bundle map covering the identity, where $\mathrm{Spin}(n)$ acts on FM via λ_n.

Spin structures II

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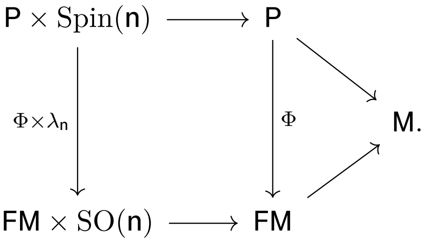

Spin structures III

Spin structures turn out not to depend on the orientation or the Riemannian metric:

Theorem

- M admits a spin structure if and only if the first two Stiefel-Whitney classes of M vanish:

$$w_1(M) = w_2(M) = 0.$$

- In this case, spin structures are classified by the first cohomology $H^1(M; \mathbb{Z}_2)$.

Spin^r structures I

What can we do with non-spin manifolds?

Idea: enlarge the spin group:

$$\begin{array}{ccc} \mathrm{Spin}(n) & \xrightarrow{\qquad \lambda_n \qquad} & SO(n). \\ & \searrow \quad \nearrow & \\ & L & \end{array}$$

Example

$$\mathrm{Spin}^{\mathbb{C}}(n) = \frac{\mathrm{Spin}(n) \times U(1)}{\langle(-1,-1)\rangle}, \qquad \mathrm{Spin}^{\mathbb{H}}(n) = \frac{\mathrm{Spin}(n) \times Sp(1)}{\langle(-1,-1)\rangle}.$$

Spin^r structures II

Note that $\mathrm{U}(1) \cong \mathrm{Spin}(2)$ and $\mathrm{Sp}(1) \cong \mathrm{Spin}(3)$.

Definition

$$\mathrm{Spin}^r(n) := \frac{\mathrm{Spin}(n) \times \mathrm{Spin}(r)}{\langle(-1,-1)\rangle}.$$

Definition

A **spin^r structure** on an oriented Riemannian n-manifold is a lift of the structure group of the positively oriented orthonormal frame bundle FM to $\mathrm{Spin}^r(n)$ along the composition

$$\begin{aligned} \lambda_n^r \colon \ \mathrm{Spin}^r(n) &\to \mathrm{SO}(n) \times \mathrm{SO}(r) \to \mathrm{SO}(n) \\ [a, b] &\mapsto (\lambda_n(a), \lambda_r(b)) \mapsto \lambda_n(a). \end{aligned}$$

Spin^r structures III

In other words, a **spin^r structure** on M consists of the following data:

- a principal $\mathrm{Spin}^r(n)$-bundle P over M, and
- a $\mathrm{Spin}^r(n)$-equivariant bundle map $\Phi\colon P \to FM$, where $\mathrm{Spin}^r(n)$ acts on FM through λ^r_n.

Definition

The rank-r vector bundle associated to P along the composition

$$\mathrm{Spin}^r(n) \to \mathrm{SO}(n) \times \mathrm{SO}(r) \to \mathrm{SO}(r)$$

is called the **auxiliary bundle** of the spin^r structure.

Characterisation

Theorem (Albanese - Milivojević, 2021 [1])

The following are equivalent for an oriented Riemannian manifold M:

1. M is spin^r;
2. there is an orientable rank-r real vector bundle $\pi : E \to M$ such that $TM \oplus E$ is spin, i.e., $w_1(TM \oplus E) = w_2(TM \oplus E) = 0$;
3. M embeds in a spin manifold with codimension r.

A few examples

Examples

1. Every oriented n-manifold M admits a spin^n structure.
   Take $E = TM$, and note that

$$w_2(TM \oplus E) = w_2(TM) + w_1(TM)w_1(E) + w_2(E) = 2w_2(TM) = 0.$$

2. Every almost-complex manifold admits a spin^2 structure.
   Take E to be the anticanonical bundle of an almost-complex structure, and compute

$$w_2(TM \oplus E) = w_2(TM) + w_2(E) = 2(c_1(TM) \bmod 2) = 0.$$

3. Every almost-quaternionic manifold admits a spin^3 structure.
   Take E to be the rank-3 subbundle of $\mathrm{End}(TM)$ spanned by I, J, K.

Proof of Albanese-Milivojević 1 $\iff$ 2

- M is spin^r $\implies$ $\exists$ E such that TM $\oplus$ E is spin:
  Take E to be the auxiliary bundle of a spin^r structure. Then, the frame bundle of TM $\oplus$ E lifts to $\mathrm{Spin}^r(n)$ along

$$\mathrm{Spin}^r(n) \to \mathrm{Spin}(n+r) \to \mathrm{SO}(n+r).$$

  In particular, it lifts to $\mathrm{Spin}(n+r)$.
- $\exists$ E such that TM $\oplus$ E is spin $\implies$ M is spin^r:
  This follows from the fact that the following is a pullback diagram in the categorical sense:

$$\begin{array}{ccc} \mathrm{Spin}^r(n) & \longrightarrow & \mathrm{Spin}(n+r) \\ \downarrow & & \downarrow \\ \mathrm{SO}(n) \times \mathrm{SO}(r) & \longrightarrow & \mathrm{SO}(n+r). \end{array}$$

Proof of Albanese-Milivojević 2 $\iff$ 3

- $\exists$ E such that $TM \oplus E$ is spin $\implies$ M embeds into a spin manifold with codimension r:

  M embeds with codimension r into the total space of E, which is spin because

$$w_2(TE) = w_2(\pi^*(TM \oplus E)) = \pi^*(w_2(TM \oplus E)) = 0.$$

- M embeds into a spin manifold with codimension r $\implies$ $\exists$ E such that $TM \oplus E$ is spin:

  Let $\iota \colon M \hookrightarrow X$ be such an embedding, and take E to be the normal bundle of ι. Then,

$$0 = \iota^*(w_2(TX)) = w_2(\iota^*(TX)) = w_2(TM \oplus E).$$

$\square$

Invariance

Let G be a connected Lie group acting smoothly on M by isometries.

Then, G acts naturally on FM by bundle isomorphisms.

Definition

A G-**invariant spin^r structure** on M is a spin^r structure (P, Φ) where both P and Φ are G-equivariant.

$$\begin{array}{ccccc} G \times P \times \mathrm{Spin}^r(n) & \longrightarrow & P & & \\ \Big\downarrow{\scriptstyle \mathrm{Id}_G \times \Phi \times \lambda^r_n} & & \Big\downarrow{\scriptstyle \Phi} & \searrow & \\ & & & & M. \\ & & & \nearrow & \\ G \times FM \times SO(n) & \longrightarrow & FM & & \end{array}$$

Homogeneous Spaces

- Let M^n be an oriented Riemannian homogeneous G-space, and fix $o \in M$.
- Then, $M \cong G/H$, where $H = \mathrm{Stab}_G(o)$.
- For every $h \in H$,

$$(L_h)_* \colon T_o M \to T_{h \cdot o} M = T_o M.$$

Definition

The **isotropy representation** is defined as

$$\begin{aligned} \sigma \colon H &\to \mathrm{SO}(T_o M) \cong \mathrm{SO}(n) \\ h &\mapsto (L_h)_*. \end{aligned}$$

The positively oriented orthonormal frame bundle FM of M is isomorphic to $G \times_\sigma \mathrm{SO}(n)$.

Invariant spin^r Structures on Homogeneous Spaces

Suppose σ lifts to $\mathrm{Spin}^r(n)$:

$$\begin{array}{ccc} & & \mathrm{Spin}^r(n) \\ & \overset{\tilde{\sigma}}{\nearrow} & \downarrow \lambda^r_n \\ H & \xrightarrow{\sigma} & \mathrm{SO}(n). \end{array}$$

Then, the pair (P, Φ) where

- $P = G \times_{\tilde{\sigma}} \mathrm{Spin}^r(n)$, and
- $\Phi \colon P \to FM, \quad [g, \mu] \mapsto [g, \lambda^r_n(\mu)]$

defines a G-invariant spin^r structure on M.

Classification

Theorem (A. - Lawn, 2023, [3])

Let G/H be an n-dimensional oriented Riemannian homogeneous space with H connected and isotropy representation $\sigma\colon H \to \mathrm{SO}(n)$. Then, there is a bijective correspondence between

- G-invariant spin^r structures on G/H modulo G-equivariant equivalence, and
- Lie group homomorphisms $\varphi\colon H \to \mathrm{SO}(r)$ such that $\sigma \times \varphi\colon H \to \mathrm{SO}(n) \times \mathrm{SO}(r)$ lifts to $\mathrm{Spin}^r(n)$ along λ^r_n modulo conjugation by an element of $\mathrm{SO}(r)$.

□

Invariant spin^r structures on spheres

Sphere	Acting group G	Minimal r for G-invariant spin^r structure
S^n	$SO(n+1)$	$r = n, \quad \text{if } n \neq 4$ $r = 3, \quad \text{if } n = 4$
S^{2n+1}	$U(n+1)$	$r = 2$
S^{2n+1}	$SU(n+1)$	$r = 1$
S^{4n+3}	$Sp(n+1)$	$r = 1$
S^{4n+3}	$Sp(n+1) \cdot U(1)$	$r = 1, \quad \text{if } n \text{ odd}$ $r = 2, \quad \text{if } n \text{ even}$
S^{4n+3}	$Sp(n+1) \cdot Sp(1)$	$r = 1, \quad \text{if } n \text{ odd}$ $r = 3, \quad \text{if } n \text{ even}$
S^6	G_2	$r = 1$
S^7	$Spin(7)$	$r = 1$
S^{15}	$Spin(9)$	$r = 1$

Special invariant spin^r structures on spheres and holonomy lifts

Theorem (A.-Lawn, 2023 [3])

Let G be the holonomy group of a simply connected irreducible non-symmetric Riemannian manifold of dimension $n + 1 \geq 4$. Let $H \leq G$ be a subgroup such that $S^n \cong G/H$, which exists, by Berger's classification. Then, the following are equivalent:

1. There exists a homomorphic lift of the holonomy representation to $\mathrm{Spin}^r(n + 1)$.
2. S^n has a G-invariant spin^r structure with strongly G-trivial auxiliary bundle. □

Spinors

The complex vector bundle $\Sigma M \to M$ associated to a spin structure via

$$\Delta_n : \mathrm{Spin}(n) \to \mathrm{End}_{\mathbb{C}}(\Sigma_n)$$

is called the **spinor bundle**: its sections are known as **spinors**.

Clifford multiplication: tangent vectors act fibrewise on spinors via, and

$$X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2g(X, Y)\psi$$

for all vector fields $X, Y \in \Gamma(TM)$ and spinors $\psi \in \Gamma(\Sigma M)$.

The Levi-Civita connection of M induces the **spin connection** ∇ on ΣM.

Generalised Killing spinors

A spinor ψ is **generalised Killing** if it satisfies

$$\nabla_X \psi = A(X) \cdot \psi,$$

for all vector fields X, where A is a symmetric endomorphism of TM.

- If $A = 0$, ψ is **parallel**;
- If $A = \lambda \, \mathrm{Id}$ for some constant $\lambda \in \mathbb{C}$, ψ is **Killing**.

The existence of these spinors is often related to curvature properties and G-structures.

Spin^r spinors

Let (P, Φ) be a spin^r structure. For odd m, the m-**twisted spin^r spinor bundle** $\Sigma^m_{n,r}M$ is the one associated to P via the representation

$$\Delta^m_{n,r} := \Delta_n \otimes \Delta_r^{\otimes m}$$

of $\mathrm{Spin}^r(n)$. Sections of $\Sigma^m_{n,r}M$ are called **spin^r spinors**.

The Levi-Civita connection on M and a connection on the auxiliary bundle determine a connection on each $\Sigma^m_{n,r}$.

There is a characterisation of special holonomy in terms of **parallel twisted pure spinors** by Herrera - Santana, 2019 [4].

Invariant spin^r spinors on projective spaces

Jointly with Hofmann [2], we obtained the following:

M	G	r	m	$\dim\left(\Sigma^{m}_{*,r}\right)_{inv}$	Special spinors	Geometry
$\mathbb{CP}^n$	$SU(n+1)$	2	1	2	pure, parallel	Kähler-Einstein
$\mathbb{CP}^{2n+1}$	$Sp(n+1)$	2, if n even	1	2	pure, parallel	Kähler-Einstein
		1, if n odd	1	2	generalised Killing	Einstein, nearly Kähler ($n = 1$)
$\mathbb{HP}^n$	$Sp(n+1)$	3	n	1	pure, parallel	quaternionic Kähler

Table: For each compact, simple, and simply connected Lie group G acting transitively on M: the minimum values of r, m such that M admits a G-invariant spin^r structure that carries a non-zero invariant m-twisted spin^r spinor, the dimension of the space of such invariant spinors, and the geometric significance of these.

References

[1] M. Albanese and A. Milivojević. “Spin^h and further generalisations of spin”. In: J. Geom. Phys. 164 (2021), Paper No. 104174, 13. ISSN: 0393-0440. DOI: 10.1016/j.geomphys.2021.104174.

[2] D. Artacho and J. Hofmann. “The geometry of generalised spin^r spinors on projective spaces”. In: SIGMA Symmetry Integrability Geom. Methods Appl. 21 (2025), Paper No. 017, 32. ISSN: 1815-0659. DOI: 10.3842/SIGMA.2025.017.

[3] D. Artacho and M.-A. Lawn. Generalised Spin^r Structures on Homogeneous Spaces. 2023. DOI: 10.48550/ARXIV.2303.05433.

[4] R. Herrera and N. Santana. “Spinorially twisted Spin structures. II: twisted pure spinors, special Riemannian holonomy and Clifford monopoles”. In: SIGMA Symmetry Integrability Geom. Methods Appl. 15 (2019), Paper No. 072, 48. ISSN: 1815-0659. DOI: 10.3842/SIGMA.2019.072.

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Thank you

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